Japanese Temple Geometry: A Digital Sangaku About a Regular Pentagon and the Golden Ratio

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Sangaku are wooden tablets depicting geometric or mathematical problems. They are objects typical of the Edo period. Since these tables were exposed in the temples, the related geometry is known as the Japanese Temple Geometry. Here we illustrate a digital approach to Sangaku. The specific problem we are discussing in the construction of a regular pentagon.

During the Edo period, it was possible to see under the roofs of Japanese shrines or temples some peculiar wooden tablets, known as "Sangaku". The peculiarity of these tablets was that they were representations of geometric and mathematical problems. These tables were specific of the Japanese culture and of the historical period of the shogunate. For being the Sangaku tables held in temples and shrines, the related geometry is defined as the Japanese Temple Geometry [1-6].

In the history of Japan, the Edo period lasted between 1603 and 1868. During it, the country was under the rule of the Tokugawa shogunate. Several economic and social features characterized this form of government; among them, we find also some isolationist foreign policies, which regarded also the culture. The shogunate ended with the Meiji Restoration in 1868.

During the period of isolation, in Japan some new arts flourished, such as the Kabuki, a form of theater known for the stylization of its drama and for the elaborate make-up of some of its performers [7,8]. Another well-known form of art that flourished in this period was the Ukiyo-e, consisting in woodblock prints and paintings of several subjects. The term Ukiyo-e can be translates as "pictures of the floating world" [9]. When Edo, that is Tokyo, became the seat of the shogunate government, the town had a rapid economic growth. In this manner, many people had the possibility of indulging in a hedonistic lifestyle, represented by the Ukiyo-e images. This art evolved from the first monochromatic prints to the Nishiki-e, "brocade prints", with a full-color production made by several woodblocks, one for each of the used colors. Sometimes embossing was used in the printing too [10]. The Ukiyo-e production went into steep decline under the modernization of the Meiji restoration [9].

Because of the large use of woodblocks for artistic printing, it is not surprising that a part of them was used also for another form of intellectual pleasure, that of the Sangaku, that is, that of the tablets used to challenge people by geometric and mathematical questions. Actually, besides being mathematical statements, the Sangaku tablets were works of art as told in the forward of the book written by Hidetoshi and Rothman on the Japanese Temple Geometry [3]. They were created using the Japanese mathematics, (wasan), developed during the Edo period in parallel to western mathematics. Many of these tablets were lost during the modernization that followed the Edo period, but around nine hundred are still existing [11].

As told in [12], Sangaku problems typically involve mutually tangent circles or tangent spheres, with specific examples including the properties of the Ajima-Malfatti points, of the Japanese theorem, and of Kenmotu point. About Ajima-Malfatti points, [13] explains that in 1803 Malfatti posed the problem of determining the three circular columns of marble of possibly different sizes which, when carved out of a right triangular prism, would have the largest possible total cross section. This is equivalent to finding the maximum total area of three circles packed inside a right triangle of any shape without overlapping [13]. The general Malfatti problem on an arbitrary triangle was actually formulated and solved earlier by the Japanese geometer Chokuen Ajima (1732-1798) [1,12,14].

Here, we want to contribute with a Sangaku about the construction of the regular pentagon and the golden ratio. Our Sangaku will be a digital one, that is, a geometric representation in a digital image, given by a matrix of pixels having grey or color tones. Let us note that a digital processing of Sangaku has been already proposed in [15].
Our starting point is the article [16]. It is told that in [1] we can find an elegant construction of the regular pentagon, due to Yosifusa Hirano, "probably a 19th century math fan". The construction has been included in a manuscript Sanpo Jyojutu Kaigi, by Chorin Kawakita (1840-1919). Kawakita wrote that his friend Hirano discovered a method of construction of regular pentagon, method which is "original, elementary and excellent" [16].
Let us consider the following diagram in Figure 1, which is like that given in [16].

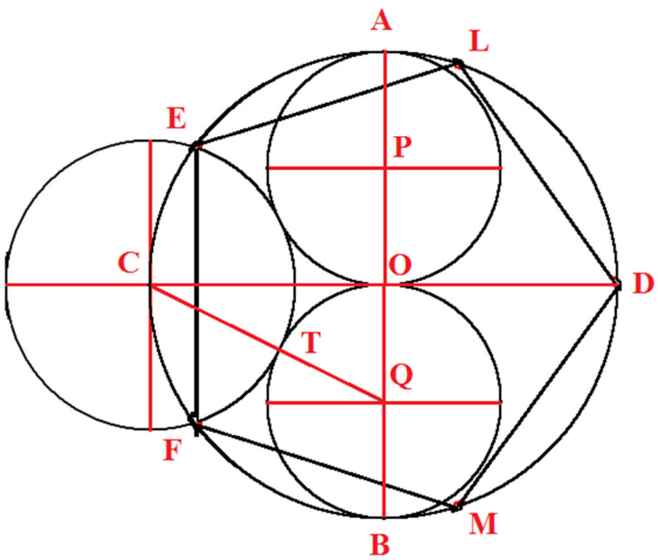

Figure 1: Hirano's construction of a regular pentagon.

In the diagram, *AB* and *CD* are perpendicular diameters of a circle with center O. The circles with centers P and Q have diameters *AO* and *BO*, respectively. The straight line *CQ* intersects the circumference with center Q in T. We can draw the circle with center C and radius *CT*. The circumference of this circle intersects the circumference having radius *AO* in E and F. These two points are two successive vertices of the regular pentagon inscribed in the circle.
Let us assume radius *OA* equal to 1. We have that [16]:

$$CQ = \sqrt{1 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$$

By construction [16]:

$$CE = CF = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5} - 1}{2}$$

Further,

$$\sin(\angle CDF) = CF / CD = \frac{\sqrt{5} - 1}{4} \tag{1}$$

Eq.(1) means that $\angle CDF = 18°$. So we have that $\angle COF = 36°$ and then $\angle EOF = 72°$. Therefore, *EF* is the side of the regular pentagon.
This construction is so simple that we can easily apply to have our digital Sangaku. Let us create it using circles that we can color with different grey tones. We start from the tablet that we paint in black. We are making this using a FORTRAN program for giving grey-tone values to the pixels of an image, the frame of which is representing the tablet. After fixing the center of the image and the two perpendicular axes x and y, we draw the white circle with radius OA with O at the center of the image. Then, we draw the two grey circles (light grey) on the diameter AOB. Using the point C as center, we find the circle (dark grey), tangent to the two light grey circles. It is then easy to find the red line, which is passing through the pixels where three colors meet (white, black and dark grey). In this manner, we have the side of the pentagon. After, it is easy to find on the circumference of the white circle, the other vertices of the regular pentagon, marked in red in the Figure 2, which is giving the resulting digital Sangaku.

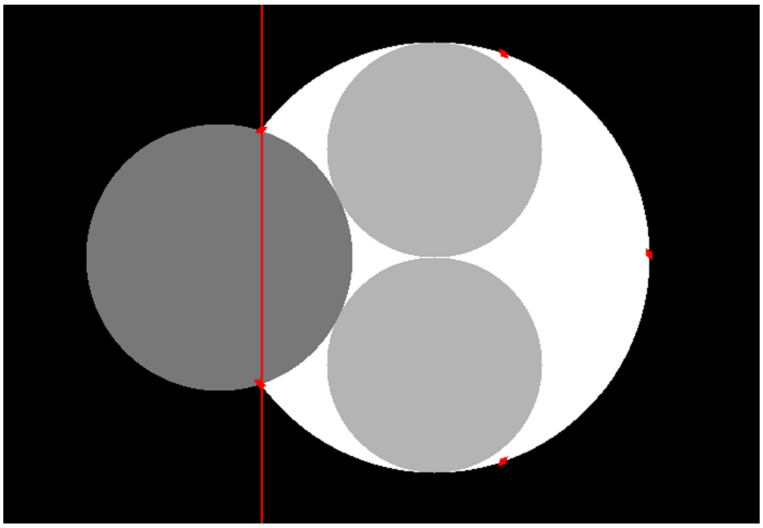

Figure 2: A digital Sangaku for constructing a regular pentagon.

The Figure 2 is the result of a FORTRAN program, which determines the circles, the red line, the side of the pentagon and the position of its vertices. If we like, we can add the other sides of the regular pentagon, by any software for drawing lines. The result is given in the Figure 3.

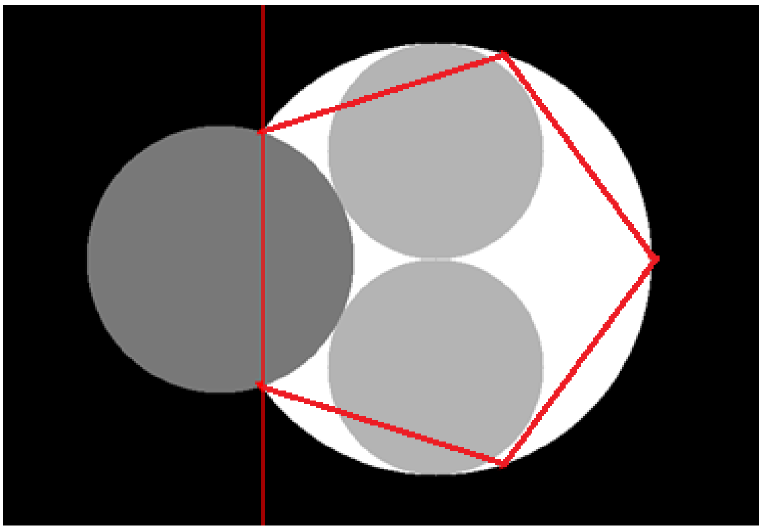

Figure 3: From the digital Sangaku of Figure 2 we can easily obtain the regular pentagon.

However, there is a further property we can add to our discussion, a property not given in [16]. Let us consider another circle, the radius of which being *OR*, as depicted in the Figure 4. We can easily find that the ratio *OR/OA* is the golden ration.

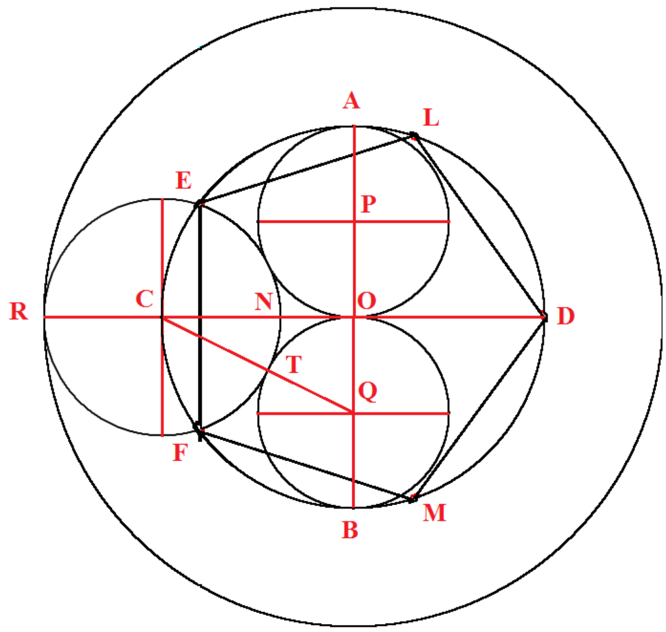

Figure 4: Let us add the circle having radius *OR*. The ratio *OR/OA* is the golden ratio.

As shown by the Figure 3, we can add another circle, that having radius *OR*. We can see that the ratio *OR/OA* is the golden ratio φ. In fact:

$$CT = CN = CE = CF = \frac{\sqrt{5}-1}{2} \; ; \; ON = OA - CN = 1 - CT = 1 - \frac{\sqrt{5}-1}{2}$$

So:

$$OR = 1 - CT + 2 \cdot CT = 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} = \varphi$$

Then, our Sangaku could be drawn as in the Figure 5.

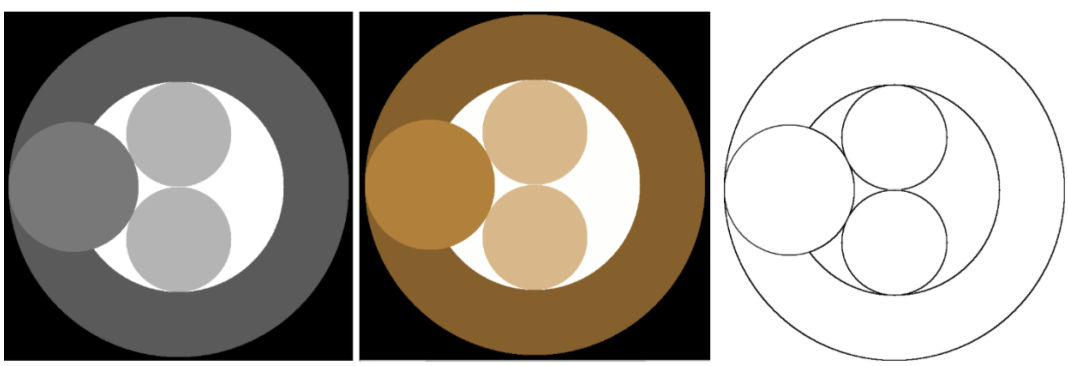

Figure 5: To the circles of the Figure 2, we can add also that which a radius equal to the golden ratio times the radius of the white circle. In the middle, a colored version is also given.

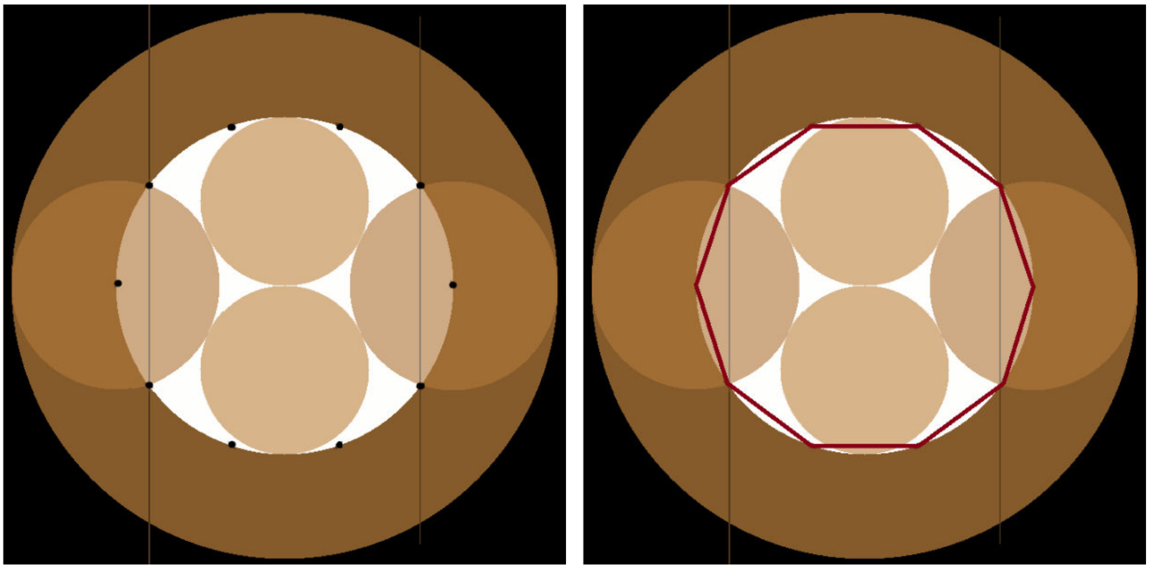

Figure 6: Using the Sangaku of the Figure 5 and its mirror symmetry, it is easy to draw a regular decagon.

Of course, we can continue and use the symmetries of the geometric figures too. Then, from the Figure 5, in which we have marked the vertices of the regular pentagon, as done in the Figure 2, we can obtain a mirror image. Combining the two images, we have the Sangaku shown in the Figure 6 on the left. It is enough to link the black dots to have the regular decagon.
Other digital Sangaku are under study.

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